Perturbational approach to the possible quantum capacity of additive Gaussian quantum channel

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Abstract

For a quantum channel with additive Gaussian quantum noise, at the large input energy side, we prove that the one shot capacity is achieved by the thermal noise state for all Gaussian state inputs. For a general case of n copies input, we show that up to first order perturbation, any non-Gaussian perturbation to the product of identical thermal states input has a less quantum information transmission rate when the input energy tends to infinitive.

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Quantum capacity is one of the main issues in quantum information theory. It is concerned with the transmission ability of unknown quantum state on a given quantum channel. The critical quantity involved in the quantum capacity is the coherent information (CI) $I_c(\sigma,\mathcal{E}) = S(\mathcal{E}(\sigma)) - S(\sigma^{QR'})$ [1] [2]. Here $S(\varrho) = -\mathrm{Tr}\varrho\log_2\varrho$ is the von Neumann entropy, σ is the input state, the application of the channel $\mathcal{E}$ results the output state $\mathcal{E}(\sigma)$; $\sigma^{QR'} = (\mathcal{E}\otimes\mathbf{I})(|\psi\rangle\langle\psi|)$, with R referred to the 'reference' system[1] (the system under process is Q system with annihilation and creation operators a and $a^\dagger$, we denote σ^Q as σ for simplicity), $|\psi\rangle$ is the purification of σ. The quantum channel capacity is[3][4][5]

$$Q = \lim_{n\to\infty}\sup_{\sigma_n}\frac{1}{n}I_c(\sigma_n,\mathcal{E}^{\otimes n}). \tag{1}$$

Quantum capacity exhibits a kind of nonadditivity [6] that makes it extremely hard to deal with. The first example with calculable quantum capacity is quantum erasure channel[7]. Other examples are dephasing qubit channel[8], amplitude damping qubit channel[9], and continuous variable lossy channel[10], where the channels are either degradable or anti-degradable[11]. Gaussian quantum channel [12] (additive classical Gaussian channel followed Holevo [13]) is quite essential in quantum information theory. Unfortunately, this channel is neither degradable nor anti-degradable[14] [15] makes the technics developed for calculating the quantum capacity unapplicable.

The quantum capacity of the Gaussian quantum channel has been conjectured as[13]

$$Q = \max\{0, -\log_2(eN_n)\}, \tag{2}$$

where N_n specifies the Gaussian quantum channel. It can be achieved by quantum error-correction codes[12]. For additive Gaussian quantum channel, we have [12] [16]

$$\mathcal{E}(\sigma) = \frac{1}{N_n}\int\frac{d^2\alpha}{\pi}\exp(-|\alpha|^2/N_n)\mathcal{D}(\alpha)\,\sigma\mathcal{D}^\dagger(\alpha), \tag{3}$$

where $\mathcal{D}(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ is the displacement operator. Any quantum state σ can be equivalently specified by its characteristic function $\chi_\sigma(\mu) = Tr[\sigma\mathcal{D}(\mu)]$, and inversely $\sigma = \int[\prod_i\frac{d^2\mu_i}{\pi}]\chi_\sigma(\mu)\mathcal{D}(-\mu)$. The characteristic function of the noisy state $\sigma' = \mathcal{E}(\sigma)$ is $\chi'_\sigma(\mu) = \chi_\sigma(\mu)e^{-N_n|\mu|^2}$.

A single mode thermal state ρ has a characteristic function of the form $\chi(\mu) = \exp[-(N+\frac{1}{2})|\mu|^2]$, and we have $\rho = \int\frac{d^2\mu}{\pi}\chi(\mu)\mathcal{D}(-\mu) = (1-v)v^{a^\dagger a}$ with $v = N/(N+1)$,[conventionally in the following, $v_x = N_x/(N_x+1)$], where N is the average photon number. The noisy state is $\rho' = \mathcal{E}(\rho) = (1-v')v'^{a^\dagger a}$, with average photon number $N' = N + N_n$, this accounts for the 'additive' of the channel.

We now consider a Gaussian state ρ_G (which comprises thermal noise state as its special case) input to the channel. A single mode Gaussian state is described by its real correlation matrix α (we drop the first moments of the state for they can be removed by local operations). We have $\alpha = \begin{bmatrix}\alpha_{qq} & \alpha_{qp}\\ \alpha_{qp} & \alpha_{pp}\end{bmatrix}$. The energy of the Gaussian state is $E = Tr[(a^\dagger a+\frac{1}{2})\rho_G] = \frac{1}{2}(\alpha_{qq}+\alpha_{pp})$. For a Gaussian state input ρ_G, the output ρ'_G and the joint output state $\rho_G^{QR'}$ are still Gaussian. The symplectic eigenvalues [13] of these states can be obtained. The coherent information is

$$I_c(\rho_G,\mathcal{E}) = g(d_0-\frac{1}{2}) - g(d_1-\frac{1}{2}) - g(d_2-\frac{1}{2}), \tag{4}$$

with

$$d_0 = \sqrt{N_n^2 + 2N_nE + E^2x} \tag{5}$$

$$d_{1,2} = \sqrt{\frac{1}{2}[N_n^2 + 2N_nE + \frac{1}{2} \pm N_nD_G]}, \tag{6}$$

where $D_G = \sqrt{(N_n+2E)^2 + 1 - (2E)^2x}$, with $x = \det(\alpha)/E^2$. Here $g(s) = (s+1)\log(s+1) - s\log s$ is the bosonic entropy function. When E is fixed, the maximum of x can be obtained with the derivatives on $x - \lambda(\alpha_{qq}+\alpha_{pp}-2E)$, where λ is the Lagrange multiplier. It follows that the maximum value $x = 1$ is achieved when

$\alpha_{qq}=\alpha_{pp}=\sqrt{E}$, $\alpha_{qp}=0$. At sufficiently large input energy E, calculating $\frac{dI_c(\rho_G,\mathcal{E})}{dx}$ and expanding the expression with E^{-1}, we then obtain

$$\frac{dI_c(\rho_G,\mathcal{E})}{dx}=\frac{1}{2Ex^2}(\frac{1}{3N_n}-2N_n)+o(\frac{1}{E^2}), \qquad (7)$$

Since $N_n<1/e$ (Otherwise $I_c=0$, this is shown by (2), also by numeric results of Gaussian input), thus $\frac{dI_c(\rho_G,\mathcal{E})}{dx}$ is positive for sufficiently large input energy. While x has its global maximum value $x=1$, so the coherent information achieves its maximum at $x=1$ which corresponds to thermal noise state input. Hence we can conclude that for sufficient large but definite input energy, the one-shot quantum information capacity of Gaussian quantum channel is achieved by thermal noise state input of all Gaussian state inputs.

For thermal state input ρ, denote the annihilation and creation operators of the 'reference' R system as b and $b^\dagger$, we have[16]

$$\begin{aligned}\rho^{QR'} &= (1-v)(1-v_n)\exp[\sqrt{v}(1-v_n)a^\dagger b^\dagger] \\ &\quad\times v_n^{a^\dagger a}(vv_n)^{b^\dagger b}\exp[\sqrt{v}(1-v_n)ab], \end{aligned}\qquad (8)$$

The state $\rho^{QR'}$ can be written as $S_2(r)(\rho_A\otimes\rho_B)S_2^\dagger(r)$, $S_2(r)=\exp[r(a^\dagger b^\dagger-ab)]$ is the two-mode squeezing operator, the squeezing parameter r is determined by $\tanh 2r=2\sqrt{N(N+1)}/(2N+N_n+1)$. ρ_A and ρ_B are two thermal states with average photon numbers N_A and N_B, respectively, where $N_{A,B}=\frac{1}{2}(D\pm N_n-1)$ with $D=\sqrt{N_n^2+2(2N+1)N_n+1}$. The coherent information will be [13][16]

$$I_c(\rho,\mathcal{E})=g(N+N_n)-g(N_A)-g(N_B). \qquad (9)$$

One of the useful formula is

$$N=N_B\cosh^2 r+(N_A+1)\sinh^2 r, \qquad (10)$$

To treat with the non-Gaussian perturbation, we need the following lemmas:

Lemma 1: $(\mathcal{E}\otimes\mathbf{I})(a^{\dagger k}\rho^{QR}a^m)=v^{-(k+m)/2}b^k\rho^{QR'}b^{\dagger m}$.

proof: With the characteristic function $\chi^{QR}$of ρ^{QR}, the lhs can be written as $\frac{1}{N_n}\int\frac{d^2\alpha}{\pi}\frac{d^4\mu}{\pi^2}\exp[-\frac{|\alpha|^2}{N_n}+\mu_1\alpha^*-\alpha\mu_1^*]$ $(a^\dagger-\alpha^*)^k\chi^{QR}(\mu)\ \mathcal{D}(-\mu)(a-\alpha)^m$. After the integral on $\mu=(\mu_1,\mu_2)$, and a displacement on α (in the ordered operator product): $\alpha\to\alpha+a;\alpha^*\to\alpha^*+a^\dagger$, the lhs can be further written as

$$\begin{aligned}&\frac{1}{N_n}\int\frac{d^2\alpha}{\pi}:\alpha^{*k}\exp[-\frac{|\alpha|^2}{v_n}+\alpha(\frac{a^\dagger}{N_n}+\sqrt{v}b)] \\ &\cdot\exp[\alpha^*(\frac{a}{N_n}+\sqrt{v}b^\dagger)-\frac{a^\dagger a}{N_n}-b^\dagger b]\alpha^m:, \end{aligned}\qquad (11)$$

where the notation $:H:$ refers to that H is in its ordered operator product, that is, all creation operators are at the left of the annihilation operators. The integral can be worked out with the formula $I=\frac{1}{K}\int\frac{d^2\alpha}{\pi}\exp[-\frac{|\alpha|^2}{K}+\alpha\sigma+\alpha^*\tau]=\exp(K\sigma\tau)$, and its derivatives

$$\frac{\partial^{k+m}I}{\partial\sigma^m\partial\tau^k}=\sum_{l=0}^{\min\{k,m\}}\binom{k}{l}\binom{m}{l}l!K^{k+m-l}\sigma^{k-l}\tau^{m-l}\exp[K\sigma\tau]. \qquad (12)$$

We have

$$\begin{aligned}(\mathcal{E}\otimes\mathbf{I})(a^{\dagger k}\rho^{QR}a^m) &=: \sum_{l=0}^{\min\{k,m\}}\binom{k}{l}\binom{m}{l}l!v_n^{k+m-l} \\ &\quad(\frac{a^\dagger}{N_n}+\sqrt{v}b)^{k-l}(\frac{a}{N_n}+\sqrt{v}b^\dagger)^{m-l}\rho^{QR'}:. \end{aligned}\qquad (13)$$

Removing the ordered notation by moving all creation operators to the left of $\rho^{QR'}$ and all annihilation operators to the right of $\rho^{QR'}$, and exchanging b with $\rho^{QR'}$according to

$$\rho^{QR'}b=[b-\sqrt{v}(1-v_n)a^\dagger]/(vv_n)\rho^{QR'}, \qquad (14)$$

exchanging $b^\dagger$ with b and further with $\rho^{QR'}$, where the formula $b^{\dagger m}b^n=\sum_{i=0}^{\min\{m,n\}}\binom{m}{i}\binom{n}{i}i!(-1)^ib^{n-i}b^{\dagger(m-i)}$ is used, after all the summation, the lemma 1 is proved.

Lemma 2: $(\mathcal{E}\otimes\mathbf{I})(a^k\rho^{QR}a^{\dagger m})=v^{(k+m)/2}b^{\dagger k}\rho^{QR'}b^m$.

proof: The lhs is $\frac{1}{N_n}\int\frac{d^2\alpha}{\pi}(a-\alpha)^k\ \{\int\frac{d^4\mu}{\pi^2}\exp[-\frac{|\alpha|^2}{N_n}+\mu_1\alpha^*-\alpha\mu_1^*]\ \chi^{QR}(\mu)\mathcal{D}(-\mu)\}(a^\dagger-\alpha^*)^m$, after the integral on μ, it is

$$\begin{aligned}&\frac{1}{N_n}\int\frac{d^2\alpha}{\pi}\exp[-\frac{|\alpha|^2}{N_n}](a-\alpha)^k \\ &\exp[\alpha a^\dagger+\sqrt{v}(a^\dagger-\alpha^*)b^\dagger]\left|00\right\rangle\left\langle 00\right| \\ &\exp[\alpha^*a+\sqrt{v}(a-\alpha)b](a^\dagger-\alpha^*)^m, \end{aligned}\qquad (15)$$

where the formula $:e^{-a^\dagger a}:=\left|0\right\rangle\left\langle 0\right|$ is used. Note that $(a-\alpha)\,e^{\alpha a^\dagger}=e^{\alpha a^\dagger}a$, thus $(a-\alpha)^k\exp[\alpha a^\dagger+\sqrt{v}(a^\dagger-\alpha^*)b^\dagger]\left|00\right\rangle=\exp[\alpha a^\dagger+\sqrt{v}(a^\dagger-\alpha^*)b^\dagger](\sqrt{v}b^\dagger)^k\left|00\right\rangle$. After the integral on α, the lemma 2 is proved.

In the single mode situation, we expand the input state ρ_ε at the vicinity of ρ, the characteristic function of the input state is $\chi_\varepsilon(\mu)=\mathrm{Tr}(\rho_\varepsilon D(\mu))=\chi(\mu)(1+\varepsilon f(\mu,\mu^*))$. The perturbation item $f(\mu,\mu^*)$ is a polynomial of μ and μ^*. Typically, this may contain (1) $|\mu|^{2n}$ ($n>1$, the $n=1$ is a Gaussian type perturbation, here we discuss single mode situation, thus n can not be confused with that appeared in (1) where it stands for the number of the modes) and (2) $c\mu^n(-\mu^*)^l+c^*\mu^l(-\mu)^{*n}$ ($n\neq l$). The first type of perturbation will contribute to the first order perturbation of the eigenvalues of ρ, while the second type of perturbation has not a first order perturbation to the eigenvalues of ρ, it can only contribute to the second order perturbation.

For the first type perturbation $|\mu|^{2n}$, We have $\chi_\varepsilon(\mu)=\chi(\mu)(1+\varepsilon|\mu|^{2n})$, thus $\rho_\varepsilon=\int\exp[-(N+\frac{1}{2})|\mu|^2](1+\varepsilon|\mu|^{2n})\mathcal{D}(-\mu)\frac{d^2\mu}{\pi}=(1+\varepsilon(-1)^n\frac{d^n}{dN^n})\rho=\rho+\varepsilon\phi$. The strict eigenvalues of ρ_ε are $\lambda_k^\varepsilon=\lambda_k+\varepsilon\phi_k$, with $\lambda_k=(1-v)v^k$ and $\phi_k=\lambda_k\xi_k$,where $\xi_k=(1-v)^n\sum_{j=0}^{\min\{n,k\}}(-1)^jn!\binom{n}{j}\binom{k}{j}N^{-j}$. ξ_k is the eigenvalue of an

operator

$$\xi(a,a^\dagger)=(1-v)^n\sum_{j=0}^{n}(-1)^j\frac{n!}{j!}\binom{n}{j}N^{-j}a^{\dagger j}a^j, \qquad (16)$$

with its eigenvector $|k\rangle$. The moments of ϕ can be calculated by $Tr(a^{\dagger l}a^l\phi)=(-1)^n\frac{d^n}{dN^n}Tr(a^{\dagger l}a^l\rho)=(-1)^n\frac{d^n}{dN^n}(l!N^l)$, thus for $l<n$, the moments are nullified. The entropy of ρ_ε can be expanded up to the second order of ε as $S(\rho_\varepsilon)=S(\rho)-\frac{1}{2}\varepsilon^2\sum_k\phi_k^2/\lambda_k+o(\varepsilon^3)$, where $Tr(\phi)=0$ and $Tr(a^\dagger a\phi)=0$ are used. We have $\sum_k\phi_k^2/\lambda_k=Tr(\xi\phi)=(1-v)^n\sum_{j=0}^n(-1)^j\frac{n!}{j!}\binom{n}{j}N^{-j}Tr(a^{\dagger j}a^j\phi)=(1-v)^nN^{-n}(n!)^2$. The calculation of the entropy of the noisy state ρ_e' is straightforward, it is

$$S(\rho_\varepsilon')=S(\rho')-\frac{1}{2}\varepsilon^2\frac{(n!)^2}{N'^n(N'+1)^n}+o(\varepsilon^3). \qquad (17)$$

The purification of ρ is $\rho^{QR}=\sum_{km}\sqrt{\lambda_k\lambda_m}\,|kk\rangle\langle mm|$. The state ρ_ε^{QR} then is expanded in ε to the linear item (the ε^2 term is less important in the large N limit). $\rho_\varepsilon^{QR}=\rho^{QR}+\varepsilon\Phi$, with $\Phi=\frac{1}{2}(\Phi_0+\Phi_0^\dagger)$, $\Phi_0=(1-v)^n\sum_{j=0}^n(-1)^j\frac{n!}{j!}\binom{n}{j}N^{-j}a^{\dagger j}b^{\dagger j}v^{j/2}\rho^{QR}$. Using lemma 1, we arrive at

$$(\mathcal{E}\otimes\mathbf{I})\rho_\varepsilon^{QR}=\rho^{QR'}+\varepsilon\Phi', \qquad (18)$$

with $\Phi'=\frac{1}{2}(\Phi_0'+\Phi_0'^\dagger)$, $\Phi_0'=\xi(b,b^\dagger)\rho^{QR'}$. The eigenstates of $\rho^{QR'}$ are $|km\rangle'=S_2(r)\,|km\rangle$ with eigenvalues $\lambda_{km}=(1-v_A)v_A^k\,(1-v_B)v_B^m$. The corresponding annihilation operators are transformed to $a'=S_2(r)aS_2^\dagger(r)=a\cosh r-b^\dagger\sinh r$, $b'=S_2(r)bS_2^\dagger(r)=b\cosh r-a^\dagger\sinh r$. The first order perturbation to the eigenvalues will be $\Phi_{km}'=\langle km|'\,\Phi'\,|km\rangle'=\lambda_{km}\,\langle km|'\,\xi(b,b^\dagger)\,|km\rangle'$. Note that

$$\begin{aligned}\langle km|'\,b^{\dagger j}b^j\,|km\rangle' &= \langle km|'\sum_{i=0}^{j}\binom{j}{i}^2 b'^{\dagger i}b'^{i}a'^{(j-i)}a'^{\dagger(j-i)}\\ &\quad\times\cosh^{2i}r\sinh^{2(j-i)}r\,|km\rangle' \qquad (19)\end{aligned}$$

We can construct an operator Ω which is the diagonal part of Φ' in the basis $|km\rangle'$, that is,

$$\begin{aligned}\Omega &= N^{-n}(N+1)^{-n}\sum_{j=0}^{n}\binom{n}{j}^2[N_B(N_B+1)\cosh^2 r]^j\\ &\quad [N_A(N_A+1)\sinh^2 r]^{n-j}\frac{\partial^n}{\partial N_B^j\partial N_A^{n-j}}\rho^{QR'}. \qquad (20)\end{aligned}$$

Where we have used the following lemma.

Lemma 3: 1. $a^{\dagger j}a^j\rho=N^j\sum_{i=0}^j\frac{j!}{i!}\binom{j}{i}(N+1)^i\frac{d^i\rho}{dN^i}$; 2. $a^ja^{\dagger j}\rho=(N+1)^j\sum_{i=0}^j\frac{j!}{i!}\binom{j}{i}N^i\frac{d^i\rho}{dN^i}$.

Proof: These two equalities can be proved with mathematical induction.

For $\rho^{QR'}$ is a direct product state in the basis $|km\rangle'$, we can treat the two new modes separately. Up to ε^2 item, the entropy will be $S(\rho_\varepsilon^{QR'})=S(\rho^{QR'})-\frac{1}{2}\varepsilon^2(\sum_{km}\Phi_{km}'^2/\lambda_{km}+\eta)+o(\varepsilon^3)$, where η is the item comes from the ε^2 item in the expansion of ρ_ε^{QR} on ρ^{QR}, this item can be omitted comparing with the main item $\sum_{km}\Phi_{km}'^2/\lambda_{km}$ for large N. Denote $\Omega_{km}=\langle km|'\,\Omega\,|km\rangle'$, and make use of Eq.(20), we have

$$\begin{aligned}\sum_{km}\Phi_{km}'^2/\lambda_{km} &= \sum_{km}\Omega_{km}^2/\lambda_{km}\\ &= (n!)^2N^{-2n}(N+1)^{-2n}\\ &\quad\times\sum_{j=0}^{n}\binom{n}{j}^{2j}B^jA^{n-j} \qquad (21)\end{aligned}$$

where Eq.(10) has been used, and $B=N_B(N_B+1)\cosh^4 r$, $A=N_A(N_A+1)\sinh^4 r$. At the limit of $N\to\infty$,we have $N'\to N$, the difference of CI between ρ_ε and ρ is

$$\begin{aligned}\lim_{N\to\infty}[I_c(\rho_\varepsilon,\mathcal{E})-I_c(\rho,\mathcal{E})] &= -\frac{1}{2}\varepsilon^2\,(n!)^2\,N^{-2n}\\ \times[1-2^{-2n}\sum_{j=0}^{n}\binom{n}{j}^2] &< 0. \qquad (22)\end{aligned}$$

If we have a linear combination of above type perturbations, see $c_1\,|\mu|^{2n_1}+c_2\,|\mu|^{2n_2}$ ($n_1<n_2$), the interference term $\sum_k\phi_{1k}\phi_{2k}/\lambda_k=(-1)^{n_2}Tr(\xi_1\frac{d^{n_2}\rho}{dN^{n_2}})=0$, the similar zero interference can be proved for the perturbation to the joint state $\rho^{QR'}$. Thus each item act separately to the coherent information. So that at the sufficiently large input energy and at the single use of the channel, the first order non-Gaussian perturbation to the input thermal state will not improve the conjectured capacity of the Gaussian quantum channel.

In the n use of the channel with an input Gaussian state ρ_n,the algebraic equations of the symplectic eigenvalues [13] are not analytically solvable. Fortunately, when the input state is a two mode squeezed thermal state, we can obtain analytical result. For the input state of $\rho_2=S_2(r_2)\rho\otimes\rho S_2^\dagger(r_2)$, where $S_2(r_2)$ is the two mode squeezing operator with real parameter r_2, the coherent information is $I_c(\rho_2,\mathcal{E}^{\otimes 2})=2\max\{0,g(d_0-\frac{1}{2})-g(d_1-\frac{1}{2})-g(d_2-\frac{1}{2})\}$, where in the expression of d_i, E should be substituted with $E/2$, now E is the total energy of the two mode state; and $x=1/\cosh^2 r_2$. The detail of the calculation will be given elsewhere. The conclusion is that the maximum coherent information is achieved by the product thermal state $\rho\otimes\rho$ when the input state is two mode squeezed thermal state at sufficiently large input energy. Also with Lagrange multiplier method, it is easy to prove that at sufficiently large input energy, for all n mode product thermal states, the product of identical thermal state will achieve the maximal of coherent information. Thus an unbalanced energy distribution among modes will not increase the total coherent information. Hence, we have proved that for all product Gaussian state inputs and all product of two mode squeezed thermal state inputs, the maximum of the coherent information is achieved by product identical thermal state $\rho_n=\rho^{\otimes n}$ for sufficient large input energy.

We now turn to the first order multi-mode perturbation. We will omit the case that the perturbation can be treated

separately for each mode. What left is the perturbation with particle exchange among modes. A typical case is $\chi_{n\varepsilon}(\mu) = \chi_n(\mu)[1 + \varepsilon(c\mu_1^{k_1}\mu_2^{k_2}\cdots\mu_n^{k_n}\mu_1^{*l_1}\mu_2^{*l_2}\cdots\mu_n^{*l_n} + c^*\mu_1^{*k_1}\mu_2^{*k_2}\cdots\mu_n^{*k_n}\mu_1^{l_1}\mu_2^{l_2}\cdots\mu_n^{l_n})]$ with $\sum_{i=1}^n k_i = \sum_{i=1}^n l_i = m$ (the requirement of first order perturbation). We may denote the perturbation as $(\mathbf{k},\mathbf{l})$, with vectors $\mathbf{k} = (k_1, k_2, \cdots, k_n), \mathbf{l} = (l_1, l_2, \cdots, l_n)$ and $\mathbf{k} \neq \mathbf{l}$. The perturbed input state is $\rho_{n\varepsilon} = \rho^{\otimes n} + \phi$. To simplify the calculation, we introduce a generation function $I_\phi(\tau,\sigma) = \int \left[\Pi_i \frac{d^2\mu_i}{\pi}\right]\chi_n(\mu)D(-\mu)\exp[\mu\cdot\tau + \mu^*\cdot\sigma]$, then

$$\phi = \left(c\frac{\partial^{2m} I_\phi(\tau,\sigma)}{\Pi_i(\partial\tau_i^{k_i}\partial\sigma_i^{l_i})} + c^*\frac{\partial^{2m} I_\phi(\tau,\sigma)}{\Pi_i(\partial\tau_i^{l_i}\partial\sigma_i^{k_i})}\right)_{\tau=\sigma=\mathbf{0}}. \quad (23)$$

The eigenspace of $\rho_{n\varepsilon}$ can be classified as subspaces according to the eigenvalue $\rho^{\otimes n}$. When the eigenvalue of $\rho^{\otimes n}$ is $\lambda_j = (1-v)^n v^j$, the basis of the subspace can be $|j_1, j_2, \cdots, j_n\rangle = |\mathbf{j}\rangle$, with $\sum_{i=1}^n j_i = j$. The action of ϕ will keep j invariant, that is, it is an operator in j subspace. In the subspace, we may specify ϕ as M_j. The eigenvalues of M_j is supposed to be $\Lambda_{j\mathbf{i}}$, then $\sum_i \Lambda_{j\mathbf{i}}^2 = Tr(M_j^2) = \sum_{\mathbf{j}}\langle\mathbf{j}|\,\phi\,|\mathbf{j}\rangle\langle\mathbf{j}|\,\phi\,|\mathbf{j}\rangle = \sum_{\mathbf{j}\mathbf{j}'}\langle\mathbf{j}|\,\phi\,|\mathbf{j}'\rangle\langle\mathbf{j}'|\,\phi\,|\mathbf{j}\rangle = \sum_{\mathbf{j}}\langle\mathbf{j}|\,\phi^2\,|\mathbf{j}\rangle$, where $\mathbf{j}'$ may not be in the j subspace. Here we have used the fact that $\langle\mathbf{j}|\,\phi\,|\mathbf{j}'\rangle = 0$ for $\mathbf{j}' \notin$the subspace of j. The perturbation to the entropy will be

$$\begin{aligned} S(\rho_{n\varepsilon}) - S(\rho^{\otimes n}) &= -\frac{1}{2}\varepsilon^2\sum_{j,\mathbf{i}}\frac{\Lambda_{j\mathbf{i}}^2}{\lambda_j} + o(\varepsilon^3) \\ &= -\frac{1}{2}\varepsilon^2 Tr(\phi^2/\rho^{\otimes n}) + o(\varepsilon^3). \quad (24)\end{aligned}$$

Where the linear term of ε is nullified by the fact that $Tr(M_j) = 0$. For

$$Tr[I_\phi(\tau,\sigma)\,I_\phi(\tau',\sigma')\,/\rho^{\otimes n}] = \exp[-\frac{\tau\cdot\sigma'}{N} - \frac{\tau'\cdot\sigma}{N+1}], \quad (25)$$

Thus

$$Tr(\phi^2/\rho^{\otimes n}) = 2\,|c|^2\,\frac{\Pi_i(k_i!l_i!)}{[N(N+1)]^m}. \quad (26)$$

In obtain Eq.(25), we first work out the operator integral of I part, which have an operator that conceal the $1/\rho^{\otimes n}$ operator. Eq.(25) exhibits that any interference item of $Tr(\phi\phi'/\rho^{\otimes n})$ type will be nullified for $(\mathbf{k},\mathbf{l}) \neq (\mathbf{k}',\mathbf{l}')$. Thus each perturbation item contributes to the entropy separately.

The perturbation to the joint QR state is more sophisticated. With almost the same routine as we do in obtaining the perturbation operator in one mode situation, we can get $\rho_{n\varepsilon}^{QR} = \rho^{QR\otimes n} + \varepsilon\Phi$. Care should be taken in obtaining the operator Φ for complex coefficient c, when c is real, there are no problem; while c is complex, there are extra phase factors in transforming the eigenbasis of ϕ into the basis of direct product of the modes. However the phase factor can be absorbed in the purification process, that is, if we have $e^{i\theta}|\mathbf{j}\rangle|\mathbf{j}\rangle$ in the purified state, it will make no difference with $|\mathbf{j}\rangle|\mathbf{j}\rangle$ in obtaining the reduced state. We thus have $\Phi = \frac{1}{2}(\Phi_0 + \Phi_0^\dagger)$, The generation function of Φ_0 is $I_{\Phi_0} = \exp\left[\frac{\sigma\cdot(\tau-\mathbf{a}^\dagger)}{N+1}\right]\exp\left[\frac{\tau\cdot\mathbf{a}}{N}\right]\rho^{QR\otimes n}$. The action of the channel then is $I'_{\Phi_0} = (\mathcal{E}\otimes\mathbf{I})I_{\Phi_0} = \exp(p\tau\cdot\mathbf{b}^\dagger)\exp[\frac{\tau\cdot\sigma}{N+1} - p\sigma\cdot\mathbf{b}]\rho^{QR'\otimes n}$ according to lemma 1 and lemma2, where $p = [N(N+1)]^{-1/2}$. The contribution to the entropy should be evaluated in the eigenbasis of $\rho^{QR'\otimes n}$. We may denote the subspace of $\rho^{QR'\otimes n}$as$|i,\mathbf{i};j,\mathbf{j}\rangle'$ which has eigenvalue $\lambda_{ij} = (1-v_A)^n(1-v_B)^n v_A^i v_B^j$. In this subspace, we denote Φ'_0 as M_{ij}, the elements of M_{ij} are $\langle i,\mathbf{i},j,\mathbf{j}|'\,\Phi'_0\,|i,\mathbf{i}';j,\mathbf{j}'\rangle'$, the sum of the square of the eigenvalue of is TrM_{ij}^2. We obtain the contribution to the entropy by first evaluating $\langle i,\mathbf{i};j,\mathbf{j}|'\,I'_{\Phi_0}\,|i,\mathbf{i}';j,\mathbf{j}'\rangle' = \langle i,\mathbf{i};j,\mathbf{j}|\,V^{\dagger\otimes n}I'_{\Phi_0}V^{\otimes n}\,|i,\mathbf{i}';j,\mathbf{j}'\rangle$, which is $\exp[\tau\cdot\sigma/(N+1)]$ $\langle i,\mathbf{i};j,\mathbf{j}|\exp[p\tau\cdot(\mathbf{b}^\dagger\cosh r + \mathbf{a}\sinh r)]\,\exp[-p\sigma\cdot(\mathbf{b}\cosh r + \mathbf{a}^\dagger\sinh r)\,|i,\mathbf{i}';j,\mathbf{j}'\rangle$. We expand the exponent of the operators to drop the terms that do not keep the total particle numbers in A and B parts respectively. Denote $I_B(\tau,\sigma) = \sum_{k=0}^\infty \frac{(-1)^k}{k!^2}(p\cosh r)^{2k}(\tau\cdot\mathbf{b}^\dagger)^k(\sigma\cdot\mathbf{b})^k$, $I_A(\tau,\sigma) = \sum_{k=0}^\infty\frac{(-1)^k}{k!^2}(p\sinh r)^{2k}(\tau\cdot\mathbf{a})^k(\sigma\cdot\mathbf{a}^\dagger)^k$. Then in the calculation of the contribution to the entropy become a trace on the whole space, the restriction on the subspace is removed. We have the generation function

$$\begin{aligned} F &= \exp[(\tau\cdot\sigma + \tau'\cdot\sigma')/(N+1)] \\ &\quad\times Tr[(I_B(\tau,\sigma)I_B(\tau',\sigma')/\rho_B^{\otimes n}] \\ &\quad\times Tr[I_A(\tau,\sigma)I_A(\tau',\sigma')/\rho_A^{\otimes n}] \\ &= \sum_{m=0}^\infty\sum_{l=0}^m\frac{B^lA^{m-l}}{[l!(m-l)!]^2[N(N+1)]^{2m}} \\ &\quad\times(\tau\cdot\sigma')^m(\tau'\cdot\sigma)^m. \quad (27)\end{aligned}$$

, with which the perturbation to the entropy can be calculated, where we have used another generation function in evaluating F, and at the final step we exchange the orders of summation and make use of Eq.(10) to conceal the factor $\exp[(\tau\cdot\sigma + \tau'\cdot\sigma')/(N+1)]$. The generation function is about that the two ingredients both come from Φ_0, if both come from $\Phi_0^\dagger$,the result will be the same. In the case intercross of Φ_0 and $\Phi_0^\dagger$, we should substitute $(\tau\cdot\sigma')^m(\tau'\cdot\sigma)^m$ in Eq.(27) by $(\tau\cdot\tau')^m(\sigma'\cdot\sigma)^m$.

$$\begin{aligned}\sum_{ij}\frac{Tr[\frac{1}{2}(M_{ij}+M_{ij}^\dagger)]^2}{\lambda_{ij}} &= 2\,|c|^2\,\frac{\Pi_i(k_i!l_i!)}{[N(N+1)]^{2m}} \\ &\quad\times\sum_{l=0}^m\binom{m}{l}^2 B^lA^{m-l}(28)\end{aligned}$$

The situation is strictly like that of the single mode case. Eq.(27) indicates that each perturbation term contributes to the entropy separately. Also, the state $\rho_\varepsilon^{QR\otimes n}$ then is expanded in ε to the linear item (the ε^2 term is less important in the large N limit).

We have shown that all first order perturbation to the input product identical thermal state can only decrease the coherent information at large input energy. In one mode

case of Gaussian input, in two mode of squeezed thermal input, and non-Gaussian perturbation of one mode as well multi-mode to thermal state input, thermal or their identical product achieve maximum of the coherent information at infinitive input energy. For all inputs of product Gaussian states, product two-mode squeezed thermal states, we have proved that the channel capacity of additive Gaussian quantum channel is described by conjectured formula (2) and achieved by product of identical thermal state. All kinds of the first order perturbations (non-Gaussian) to the product of identical thermal state input will lead to a less coherent information for sufficient large input energy. Whether a bigger coherent information can be achieved by a state which is far from the product of identical thermal state still remains open.

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References

[1] B. Schumacher, Phys. Rev. A **54,** 2614 (1996); B. Schumacher and M. A. Nielsen, Phys. Rev. A **54,** 2629 (1996).

[2] S. Lloyd, Phys. Rev. A **55,** 1613 (1997).

[3] I. Devetak, IEEE Trans. inf. Theory **51**, 44(2005); I. Devetak and A. Winter, Proc. R. Soc. Lond. A, **461**, 207 (2005).

[4] H. Barnum, M. Knill and M. A. Nielsen, IEEE Trans. Inf. Theory, **46,** 1317 (2000).

[5] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Rev. Lett. **85**, 433 (2000).

[6] D. P. DiVincenzo, P. W. Shor and J. A. Smolin, Phys.Rev. A **57**, 830 (1998).

[7] C. H. Bennett, D. P. Divincenzo and J. A. Smolin, Phys. Rev. Lett. **78**, 3217 (1997).

[8] I. Devetak, and P. W. Shor, Comm. Math. Phys. **256**, 287(2005).

[9] V. Giovannetti, R. Fazio, Phys. Rev. A, **71**, 032314 (2005).

[10] M. M. Wolf, D. P¡äerez-Garc¡äia, G. Giedke, Phys. Rev. Lett. **98**, 130501(2007).

[11] F. Caruso, V. Giovannetti, Phys. Rev. A, **74,** 062307(2006).

[12] J. Harrington, and J. Preskill, Phys.Rev. A **64,** 062301 (2001).

[13] A. S. Holevo and R. F. Werner, Phys.Rev. A **63**, 032312 (2001).

[14] A. S. Holevo, Eprint, quant-ph/0607051.

[15] F. Caruso, V. Giovannetti, A. S. Holevo, New J. Phys. **8,** 310(2006).

[16] X. Y. Chen and P. L. Qiu, Chinese Physics, **10**, 779 (2001).